$$\Rightarrow \lim_{\lambda\to\infty} \sum_{n=1}^{\lambda} (a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx \rightarrow \text{converges}$$

$$\Rightarrow \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \text{ converges}$$

$\because$ 'f' is integrable on $[-\pi, \pi]$
$\Rightarrow f^2$ is also integrab
hence $\int f^2$ is converg

Theorem 3 :- Riemann-Lebesgue Theorem

If a func $\varphi(x)$ is bounded and integrable on interval $[a, b]$ and $A_n = \int_a^b \varphi(x) \cos nx\,dx$

$B_n = \int_a^b \varphi(x) \sin nx\,dx$, then $A_n \to 0$, $B_n \to 0$ as $n \to \infty$

Main Theorem :- If a function 'f' is bounded periodic with period 2π, integrable and piecewise monotonic on $[-\pi, \pi]$, then

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin x)$$

$$= \begin{cases} \frac{1}{2}[f(x-) + f(x+)] ; & -\pi < x < \pi \\ \frac{1}{2}[f(\pi-) + f(\pi+)] ; & x = \pm\pi \end{cases}$$

; a_n & b_n are F.C. of 'f'

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Some Important Questions based on above topics

1. Expand $f(x) = x\sin x$, $0 < x < 2\pi$ as a Fourier Series.

Sol^n:- We have $f(x) = x\sin x$ —— ①

We know that Fourier series for f

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \text{—— } (*)$$

For this we have to find Fourier Coefficients i.e Euler's Formula, we have.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx$$

$$\boxed{\because f(x) = x\sin x}$$

$$= \frac{1}{\pi}\left[x(-\cos x)\Big|_0^{2\pi} - \int_0^{2\pi} -\cos x\,dx\right]$$

$$= \frac{1}{\pi}\left[-2\pi\cos 2\pi + 0\right] + \underbrace{[\sin x]_0^{2\pi}}_{0}$$

$$= \frac{-2\pi}{\pi} = -2$$

$$\therefore \boxed{a_0 = -2}$$

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$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left(\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right)\Bigg|_0^{2\pi} - \int_0^{2\pi} 1\left(-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right)dx\right]$$

$\downarrow$ 0

$$= \frac{1}{2\pi}\left[2\pi\left(-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right)\right]$$

$$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{-\cancel{n}+1+\cancel{n}+1}{n^2-1} = \frac{2}{n^2-1} \quad ; n \neq 1$$

$$\therefore \boxed{a_n = \frac{2}{n^2-1} \quad ; n \neq 1}$$

Suspected point is $n = 1$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\,dx.$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right)\Bigg|_0^{2\pi} - \int_0^{2\pi} 1\left(-\frac{\cos 2x}{2}\right)dx\right.$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi\cos 4\pi}{2}\right] + \frac{1}{4\pi}\int_0^{2\pi}\cos 2x\,dx$$

$\longrightarrow 0.$

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$$\Rightarrow a_1 = \frac{1}{2\pi}[-\pi] = -\frac{1}{2}$$

$$\therefore \boxed{a_1 = -\frac{1}{2}}$$

$$\boxed{b_n = 0 \quad ; n \neq 1}$$ As function is Even. $b_n = 0$

Particular $n = 1$, then

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin^2 x \, dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \left(\frac{1-\cos 2x}{2}\right) dx = \frac{1}{2\pi}\left[\left(x^2 - \frac{x\sin 2x}{2}\right)_0^{2\pi}\right.$$

$$\left. - \int_0^{2\pi} 1\left(x - \frac{\sin 2x}{2}\right) dx\right]$$

$$= \frac{1}{2\pi}\left[(2\pi)^2 - 0\right] - \frac{1}{2\pi}\left[\frac{x^2}{2} + \frac{\cos 2x}{2}\right]_0^{2\pi}$$

$$= 2\pi - \frac{1}{\cancel{2}\pi}\left[\frac{\cancel{4}\pi^2}{\cancel{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}}\right] = \pi$$

$$\therefore \boxed{b_1 = \pi}$$

Using the values of a_0, a_1, b_1, a_n, b_n in (A)

$$x\sin x = \frac{a_0}{2} + a_1\cos x + b_1\sin x + \sum_{n=2}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= -\frac{2}{2} - \frac{1}{2}\cos x + \pi\sin x + \sum_{n=2}^{\infty} a_n\cos nx + 0$$

$$= -1 - \frac{1}{2}\cos x + \pi\sin x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx.$$

$$= -1 + \pi\sin x - \frac{1}{2}\cos x + \frac{2}{2^2-1}\cos 2x + \frac{2}{3^2-1}\cos 3x$$
$$+ \frac{2}{4^2-1}\cos 4x + \cdots$$

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2. Expand $f(x) = e^{-x}$, $0 < x < 2\pi$ as a Fourier Series

Sol^n:- We have $f(x) = e^{-x}$ —— (1)

We know that Fourier series for any fun^c.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \quad \text{—— } (*)$$

& $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$

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Now $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left[-e^{-x}\right]_0^{2\pi}$

$$= -\frac{1}{\pi}\left[e^{-2\pi}-1\right] = \frac{1-e^{-2\pi}}{\pi}$$

$$\therefore \boxed{a_0 = \frac{1-e^{-2\pi}}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx.$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{(-1)^2+n^2}\left(-1\cos nx + n\sin nx\right)\right]_0^{2\pi}$$

$$\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{n^2+1}\left(n\sin nx - \cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi}}{n^2+1}\left(0-\cos 2n\pi\right) - \frac{e^{-0}}{n^2+1}\left(0-\cos n.0\right)\right]$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{-2\pi}(-1)+1\right] = \frac{1}{\pi(n^2+1)}\left[1-e^{-2\pi}\right]$$

$$\therefore \boxed{a_n = \frac{1-e^{-2\pi}}{\pi(n^2+1)}}$$

Also $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{(-1)^2+n^2}\left(-1.\sin nx - n\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{-1}{\pi}\left[\frac{e^{-x}}{n^2+1}\left(+\sin nx + n\cos nx\right)\right]_0^{2\pi}$$

$\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$

$$= \frac{-1}{\pi(n^2+1)}\left[e^{-2\pi}\left(0+n\cos 2n\pi\right) - e^{-0}\left(0+n\cos n.0\right)\right]$$

$$= \frac{-1}{\pi(n^2+1)}\left[e^{-2\pi}(n) - n\right] = \frac{-n}{\pi(n^2+1)}\, e^{-2\pi}-1$$

$$\therefore \boxed{b_n = \left(\frac{n}{n^2+1}\right)\left(\frac{1-e^{-2\pi}}{\pi}\right)}$$

Using the values of a_0, a_n, b_n in eq^n ⊛, we get

$$f(x) = e^{-x} \sim \frac{1-e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty}\left(\frac{1-e^{-2\pi}}{\pi(n^2+1)}\cos nx + \frac{(1-e^{-2\pi})n}{\pi(n^2+1)}\sin nx\right)$$

$$= \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \left(\sum_{n=1}^{\infty}\frac{\cos nx}{n^2+1} + \sum_{n=1}^{\infty}\frac{n\sin nx}{n^2+1}\right)\right.$$

$$= \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \left(\frac{1}{2}\cos x + \frac{1}{5}\cos 2x + \frac{1}{10}\cos 3x + \cdots\right)\right.$$
$$\left. + \left(\frac{1}{2}\sin x + \frac{2}{5}\sin 2x + \frac{3}{10}\sin 3x + \cdots\right)\right]$$

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